DOCUMENT RESUME

ED 218 123 SE 038 233

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TITLE	The Diffusion of Innovation in Family Planning. Applications of First-Order Difference Equations to American Politics. [and] The Growth of Partisan Support, I: Model and Estimation, II: Model Analytics. Applications of First Order Quadratic Difference Equations to American Politics. Modules and Monographs in Undergraduate Mathematics and Its Applications Project. UMAP Units 303-305
INSTITUTION	Education Development Center, Inc., Newton, Mass.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	79
GRANT	SED-76-19615; SED-76-19615-A02
NOTE	82p.
EDRS PRICE	MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS	*College Mathematics; *Family Planning; Higher Education; Instructional Materials; *Learning Modules; *Mathematical Applications; Mathematical Models; Models; *Political Science; *Politics; Supplementary Reading Materials

ABSTRACT

This document includes three units on applications of first-order difference equations to American politics. The first module is designed to help the user: 1) develop flexibility in analyzing difference equations in quadratic format; 2) understand how Choundy and Phillips' Theorem can be used to provide information about the time path of a difference equation in a quadratic format; and 3) understand the process through which national governments adopted family planning policies. The second unit is focused on: 1) introduction of non-linear representation (first order quadratic difference equation) for political mobilization processes; 2) estimation of model parameters and substantive interpretations; and 3) investigating analytic consequences of substantive assumptions. The final module is a continuation of the second, and looks at the analytic properties of the model presented there. The unit investigates mathematical properties of first-order quadratic difference equation equilibria, local stability, and global stability, and shows ways to use these analytic results to better understand political mobilization processes. Each unit contains exercises, with answers given at the conclusion of each module. (MP)

umap

UNIT 303

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT

THE DIFFUSION OF INNOVATION IN FAMILY PLANNING

by Kathryn Newcomer Harmon

"Tremendous change has taken place in public policies concerning birth control both across nations and within nations during the last twenty years. But to characterize this change as a revolution tends to overstate the speed with which governments innovate in the area of birth control. Despite great differences in birth rates, governments have initiated the same types of contraceptive policies."

APPLICATIONS OF FIRST-ORDER DIFFERENCE EQUATIONS TO AMERICAN POLITICS

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THE DIFFUSION OF INNOVATION IN FAMILY PLANNING

by

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Intermodular Description Sheet: *UMAP Unit 303*

Title: THE DIFFUSION OF INNOVATION IN FAMILY PLANNING

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Reveiw Stage/Date: III 10/10/78

Classification: APPL FIRST ORD DIF EQUA/AMER POL

Suggested Support Materials:

References: See Section 5 of text.

Prerequisite Skills:

1. Be able to rearrange terms to transform difference equations into standard quadratic format.
2. Be able to use a theorem to analyze difference equations in standard quadratic format.

Output Skills:

1. Develop flexibility in analyzing difference equations in quadratic format.
2. Understand how Chaundy and Phillips' Theorem can be used to provide information about the time path of a difference equation in quadratic format.
3. Understand the process through which national governments adopted family planning policies.

Other Related Units:

Exponential Models of Legislative Turnover *(Unit 296)*
The Dynamics of Political Mobilization I *(Unit 297)*
The Dynamics of Political Mobilization II *(Unit 298)*
Public Support for Presidents I *(Unit 299)*
Public Support for Presidents II *(Unit 300)*
Laws That Fail I *(Unit 301)*
Laws That Fail II *(Unit 302)*
Growth of Partisan Support I *(Unit 304)*
Growth of Partisan Support II *(Unit 305)*
Discretionary Review by Supreme Court I *(Unit 306)*
Discretionary Review by Supreme Court II *(Unit 307)*
What Do We Mean By Policy? *(Unit 310)*

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MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

This unit was presented in preliminary form at the Shambaugh Conference on Mathematics in Political Science Instruction held December, 1977 at the University of Iowa. The Shambaugh fund was established in memory of Benjamin F. Shambaugh who was the first and for forty years served as the chairman of the Department of Political Science at the University of Iowa. The funds bequeathed in his memory have permitted the department to sponsor a series of lectures and conferences on research and instructional topics. The Project would like to thank participants in the Shambaugh Conference for their reviews, and all others who assisted in the production of this unit.

This material was prepared with the support of National Science Foundation Grant No. SED76-19615. Recommendations expressed are those of the author and do not necessarily reflect the views of the NSF, nor of the National Steering Committee.

THE DIFFUSION OF INNOVATION IN FAMILY PLANNING

INTRODUCTION

> About one-third of the world's couples now use some form of contraceptive as the result of a revolution in government attitudes toward birth control, the Population Reference Bureau said today.
> Dorothy Nortman of the Population Council's Center for Policy Studies, in a report to the bureau, said birth control was being used at a "level unprecedented in human history."
> The increase can be ascribed to an "180-degree reversal" in popluation policies by national governments, she said, "from almost universal indifference or condemnation of birth control only a generation ago to almost universal approval today." (New York Times, September 11, 1977.)

Tremendous change has taken place in public policies concerning birth control both across nations and within nations during the last twenty years. But to characterize this change as a revolution tends to overstate the speed with which governments innovate in the area of birth control. Despite great differences in birth rates, governments have initiated the same types of contraceptive policies. Governments have liberalized restrictions on abortion, and on the dissemination of and advertisement of contraceptive devices, and have funded the provision of contraceptive information and devices to their populations. This last type of action, the family planning policies, has constituted the predominant governmental approach (Rogers, 1973).

Everett Rogers has suggested that governmental policy-makers have been influenced greatly by the precedents set by other governments when considering the adoption of specific family planning programs or policies (Rogers, 1973). An examination of the number of countries which adopted family planning programs or policies each year in the last two decades can help in assessing the plausibility of Rogers' interpretation. That is, is it plausible that innovation in contraception spread among governmental adopters through a diffusion, or learning, process?

Figure 1 shows the number of national governments in the developing world adopting such programs in each year between 1960 and 1976, as well as the cumulative curve for the total number of adopters in each time period. It is clear that after 1964 a takeoff of sorts occurred. From 1964 onward the number of adopters increased fairly rapidly and consistently until it leveled off around 1974. The curve representing the cumulative number of adopters is somewhat S-shaped, as one would expect if a learning process was underlying governmental innovation.

The shape of the curve representing the cumulative number of adopters is supportive of Rogers' interpretation that the observation by governmental policy-makers that more and more national governments were establishing family planning programs itself stimulated innovation. Whether there were certain "pioneers" who provided cues to the other governments regarding innovation or not, the S-shaped curve supports the hypothesis that a diffusion, or learning

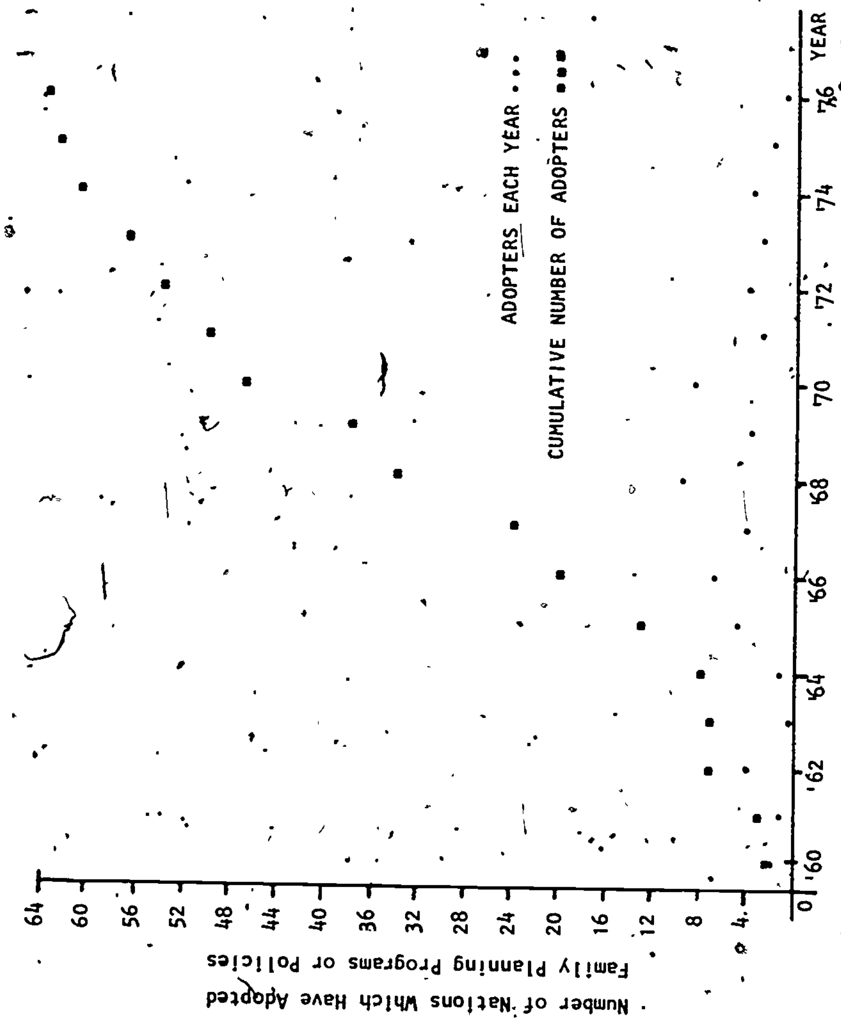


Figure 1. The number of nations adopting family planning programs or policies each year, and the cumulative number of adopters, 1960-1976.

process among the universe of potential adopters was in operation (Gary, 1973; Walker, 1969, 1973).

If we accept the diffusion interpretation as plausible it is possible to reconstruct this process with a mathematical equation which represents the interaction between governments. Through this exercise we can then use the equation to analyze the behavior of governments in shifting "from almost universal indifference or condemnation of birth control only a generation ago to almost universal approval today." In this analysis we can answer two questions: First, does the equation incorporating the notion of diffusion actually correspond to the empirical information about the adoption of family planning policies? And secondly, how many more countries are likely to adopt such policies?

1. A FORMAL REPRESENTATION OF THE DIFFUSION OF INNOVATION IN FAMILY PLANNING

If a diffusion process is in operation, as more governments adopt family planning policies, the effect on non-adopters increases. Examples are set by nations facing similar demographic and economic pressures, and information about program successes spreads to governments which have not yet adopted.

This interactive process may possibly be modeled by

$$\Delta P_t = -gP_{t-1} + fP_{t-1}(L - P_{t-1}) \qquad (1)$$

where	ΔP_t	represents the change in the number of governments which have adopted a program between time t and time t-1, i.e., $\Delta P_t = P_t - P_{t-1}$.
	L	represents the maximum number of potential adopters,
	$L - P_{t-1}$	represents the pool of potential adopters at time t,
	g	represents the loss rate whereby some governments discontinue programs adopted previously,
and	f	represents the coefficient of diffusion stemming from interaction.

In some policy areas the limit, L, may be set equal to 100% of the universe of governments, since all governments could act. In birth control, however, it is possible that religious pressures might prohibit some governments from ever acting, the limit might then be treated as a constant to be estimated.

Equation (1) can be rewritten so that the output is the cumulative proportion of governments that have adopted a family planning policy, which is a function of the proportion of the governments which have retained the program since time t-1, and some proportion of the pool of nonadopters which is affected by the diffusion process. Due to the possibility that some governments might not have retained the program, the loss rate, g, has been utilized. We can let another coefficient, e, represent the proportion of previous adopters maintaining their programs, thus

e = 1-g. Expanding ΔP_t and adding P_{t-1} to both sides of the equation results in the following:

$$P_t = P_{t-1} - gP_{t-1} + fP_{t-1}(L - P_{t-1}) \qquad (2)$$

or

$$P_t = eP_{t-1} + fP_{t-1}(L - P_{t-1}). \qquad (3)$$

When governments which adopt policies or programs are not likely to terminate them, the first coefficient, e, may be set equal to 1.0, since the loss rate, g, is zero. But recent work on the process of diffusion has highlighted the importance of this factor in analyzing diffusion curves, and indicated that this constant, too, merits empirical estimation (Eyestone, 1977).

Through expansion, and rearrangement of terms, Equation (3) begins to take on the quadratic form

$$P_t = eP_{t-1} + fL\,P_{t-1} - fP_{t-1}^2 \qquad (4)$$

or, where e = 1.0,

$$P_t = (1 + fL)\,P_{t-1} - fP_{t-1}^2. \qquad (5)$$

This equation reflects the assumption that governmental policy-makers are responding chiefly to the precedents and experience of other governments. However, in some specific policies adopted with regard to the general problem of unregulated population growth, other factors may be quite significant. Other factors could be represented in this equation by the inclusion of additional terms.

One example of the significant role which an external factor can play, along with the diffusion factor, is the initiative taken by a central governmental policy-maker in a federal governmental system. This is illustrated by the impact which U.S. Supreme Court decisions have had upon state abortion policies in the United States.

By January of 1973, when the U.S. Supreme Court handed down the key decisions which legalized abortion on demand during the first trimester of gestation, (Roe v. Wade (410 U.S. 113) and Doe v. Bolton (410 U.S. 179)) eighteen states had liberalized the previous prohibition of abortions. After the court acted on this controversial issue a flurry of state legislative activity was initiated to either facilitate the implementation of abortion services, or to secure the freedom of medical personnel in refusing to participate in such services, i.e., institutional and individual conscience laws. In 1973 thirty-nine bills were passed in state legislatures which dealt with abortion, and this number was nineteen in 1974, fifteen in 1975, and twelve in 1976. The Supreme Court initiative served as a stimulus to the state governmental policy-makers just as the legislation introduced in other state legislatures during this period influenced activity by those legislatures which had not yet acted. In this example, a coefficient, h, could represent the stimulus of the Supreme Court decisions, changing the equation to the form

$$P_t = (e + fL)P_{t-1} - fP_{t-1}^2 + h \qquad (6)$$

or, where e = 1,

$$P_t = (-f)P_{t-1}^2 + (1 + fL)P_{t-1} + h. \qquad (7)$$

This equation is a difference equation in quadratic form with the three coefficients A, B, and C corresponding to the real numbers (-f), (1 + fL) and h. This equation

differs only slightly from Equation (5) in that h is no longer set equal to zero.

2. ANALYSIS OF DIFFERENCE EQUATIONS IN QUADRATIC FORM

To aid in an analysis of the diffusion process operating as governments adopt family planning programs, difference equations with quadratic form can be analyzed to provide information about the stability and results of the diffusion process. Chaundy and Phillips have produced a theorem on conditions of convergence and divergence, and ultimate qualitative behavior which can facilitate this type of analysis (Chaundy and Phillips, 1936). John Sprague had adapted Chaundy and Phillips' work, and the applicable tests (Sprague, 1969).

Given a difference equation of the form

$$Y_t = AY_{t-1}^2 + BY_{t-1} + C, \qquad (8)$$

Chaundy and Phillips define a quantity K which is used to relate initial conditions to any equilibrium reached. K is given by

$$K = \frac{-1 \pm \sqrt{1 + 4\left(\left(\frac{B}{2}\right)^2 - \frac{B}{2} - AC\right)}}{-2} = \frac{1 \pm \sqrt{1 + 4\left(\left(\frac{B}{2}\right)^2 - \frac{B}{2} - AC\right)}}{2} \qquad (8a)$$

where A, B, and C are as in Equation (8). Also, note that if C = 0, K reduces to

$$\frac{1 \pm |1 - B|}{2}$$

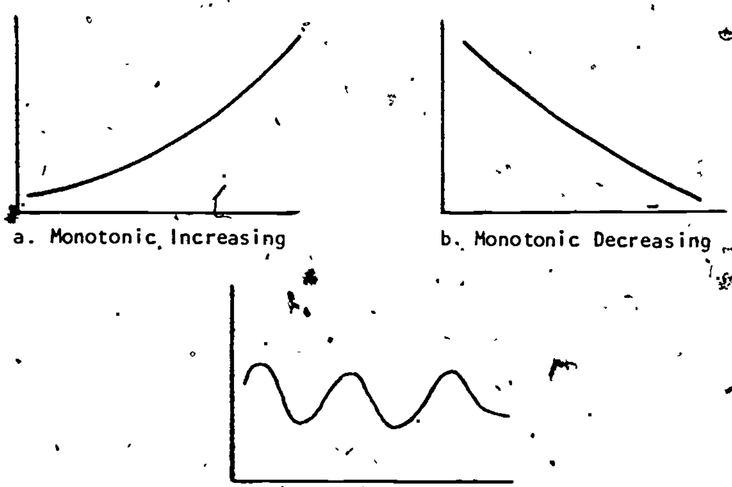
EXERCISE 1. Find K for the following difference equations:

(a) $Y_t = Y_{t-1}^2 + .2Y_{t-1}$,

(b) $Y_t = -.2Y_{t-1}^2 + .2Y_{t-1}$,

(c) $Y_t = -.2Y_{t-1}^2 + .2Y_{t-1} + (-.2)$.

Chaundy and Phillips utilize the quantity K and the initial value of an output sequence, Y_0, to identify several conditions which describe the behavior of an equation, given values for coefficients A, B, and C, and the initial condition. The conditions describe several types of time paths. A monotonic path either descreases or increases in value continuously, without deviation, and oscillation refers to alternate increases and decreases in the time path (see Figure 2).



a. Monotonic Increasing

b. Monotonic Decreasing

c. Oscillation

Figure 2. Examples of Possible Time Paths.

The conditions are as follows:

I. If K, given by Equation (8a) is not real, then Y_n diverges to infinity.

If there are real K, take the larger value for K (which must exceed 1/2)

II. If $|AY_0 + \frac{B}{2}| > K$, then Y_n diverges to infinity.

III. If $|AY_0 + \frac{B}{2}| = K$, then Y_n is stationary, although this does not mean it will converge if displaced.

IV. If $|AY_0 + \frac{B}{2}| < K$, and $1/2 \leq K \leq \frac{3}{2}$, then Y_n converges to a value

$$Y^* = \frac{1 - K - \frac{B}{2}}{A}$$

Note: This limit thus depends on Y_0, A, B, and C since K depends on C.

Note: Convergence is monotonic if $1/2 \leq K \leq 1$.

V. If $|AY_0 + \frac{B}{2}| < K$ and $\frac{3}{2} < K \leq 2$, then Y_n oscillates finitely.

VI. If $|AY_0 + \frac{B}{2}| < K$ and $K > 2$, then Y_n goes to infinity unless Y_0 is chosen so that the expression $|AY_0 + \frac{B}{2}|$ is an element of a set involving the square roots of the expression $K^2 - K$, in which case Y_n oscillates finitely. Essentially this involves solving the difference equation backwards by taking roots until Y_0 is reached.

EXERCISE 2. Identify which condition characterizes the behavior of the solution of each of the equations in Exercise 1, given an initial value Y_0 of .1 for all three equations.

3. APPLICATIONS OF THE CHAUNDY AND PHILLIPS THEOREM

Before returning to an analysis of the diffusion of family-planning programs across nations between 1960 and 1976, several applications of the Chaundy and Phillips theorem can be analyzed by setting real values for the coefficients in the diffusion equation presented earlier,

$$P_t = (-f)P_{t-1}^2 + (e + fL)P_{t-1} + h. \qquad (8)$$

Now that this equation is in standard quadratic form, by substitution the following equations hold: $A = -f$, $B = e + fL$, and $C = h$.

In the first example, assumtpions about the limit, L, and the third coefficient, h, will correspond to those underlying Equation (2). Thus, $L = 1.0$ and $h = 0$. Further, given $f = .2$, $e = 1.0$, the first equation to be analyzed is as follows:

$$Y_t = -(.2)Y_{t-1}^2 + (1.0 + (.2)\ (1.0)Y_{t-1} + 0 \qquad (9)$$

or

$$Y_t = -(.2)Y_{t-1}^2 + 1.2Y_{t-1} \qquad (10)$$

Utilizing the Chaundy and Phillips theorem, K is identified here as follows:

$$K = \frac{-1 \pm \sqrt{1 + 4((\frac{1.2}{2})^2 - \frac{1.2}{2} - (-.2*0))}}{-2}$$

or

$$K = \frac{-1 \pm \sqrt{1 - .96}}{-2} = \frac{-1 \pm .2}{-2} = .4 \text{ or } .6.$$

When we take the $K \geq .5$, and setting the initial value Y_0 equal to .05, the following calculations result:

$$|AY_0 + \frac{B}{2}| = |(-.2)\ (.05) + \frac{1.2}{2}| = .59.$$

Since $.59 < .6$, Condition IV holds, and Y_n converges to $\frac{1 - K - \frac{B}{2}}{A}$, or $\frac{1 - .6 - \frac{1.2}{2}}{-.2} = 10$, which is expected since the limit was initially set equal to 1.0. The speed with which the output approaches the limit is of interest, and Table I shows the values for the first sixteen time periods. Further examples can help illustrate the way different values for the coefficients e, f, and h, and the limit, L, affect the time path of Y_n.

TABLE 1

Values of Output from Equation (10)

$$Y_t = (-.2)Y_{t-1}^2 + 1.2Y_{t-1} + 0$$

where Y_0 = .05 and the Limit = 1.0.

Time Period	Value of Y_t
0	.05
1	.0595
2	.0707
3	.0838
4	.0992
5	.1171
6	.1377
7	.1615
8	.1886
9	.2192
10	.2534
11	.2912
12	.3325
13	.3769
14	.4239
15	.4727
16	.5226

A second example can be analyzed which differs from the first in only one respect--the value of the limit is changed. In this second case, the limit is changed from 1.0 to .8. Thus,

$$Y_t = -(.2)Y_{t-1}^2 + (1.0 + (.2)(.8))Y_{t-1} + 0 \qquad (11)$$

or

$$Y_t = -(.2)Y_{t-1}^2 + (1.16)Y_{t-1} \qquad (12)$$

Here A = -.2, B = 1.16, and C = 0. We find

$$K = \frac{-1 \pm \sqrt{1 + 4\left(\left(\frac{1.16}{2}\right)^2 - \frac{1.16}{2} - 0\right)}}{-2}$$

or

$$K = \frac{-1 \pm \sqrt{1 - .974}}{-2} = \frac{-1 \pm .16}{-2} = .42 \text{ or } .58.$$

We take $K \geq .5$, and again setting the initial value Y_0 equal to .05, the following results are obtained:

$$\left|AY_0 + \frac{B}{2}\right| = \left|(-.2)(.05) + \frac{1.16}{2}\right| = .57.$$

Since .57 < K, Condition IV again applies, and Y_n converges to $\dfrac{1 - .57 - \frac{1.16}{2}}{-.2} = .75$. Table 2 displays the values of the output of Equation (11), and it is clear that the time path approaches the limit more slowly than the time path for Equation (9) approaches the specified limit.

TABLE 2

Values of Output from Equation (12)

$$Y_t = (-.2)Y_{t-1}^2 + 1.16Y_{t-1} + 0$$

where $Y_0 = .05$ and the Limit = .8.

Time Period	Value of Y_t
0	.05
1	.0575
2	.0660
3	.0757
4	.0867
5	.0990
6	.1120
7	.1285
8	.1457
9	.1648
10	.1857
11	.2086
12	.2332
13	.2597
14	.2877
15	.3172
16	.3478

A second modification of the first example, Equation (9), illustrates the impact of differing values for the coefficients upon the time path of the output. In this third example the only change from the previous Equation (11) is that the third coefficient, h, is assigned a real value other than zero. This equation is

$$Y_t = -(.2)Y_{t-1}^2 + (1.0 + (.2)(.8))Y_{t-1} - .1 \qquad (13)$$

or

$$Y_t = -(.2)Y_{t-1}^2 + 1.16Y_{t-1} - .1 \qquad (14)$$

Here A = -.2, B = 1.16, and C = -.1. We have

$$K = \frac{-1 \pm \sqrt{1 + 4((\frac{1.16}{2})^2 - \frac{1.16}{2} - (-.2)(-.1))}}{-2}$$

or

$$K = \frac{-1 \pm \sqrt{1 - 1.044}}{-2} = \frac{-1 \pm \sqrt{-.044}}{-2}.$$

Since a negative number appears under the radical in this example, K is not a real number. Thus Condition I holds and Y_n diverges to infinity. Even though a limit was specified in this example, the inclusion of a third factor, represented by the coefficient h, caused the time path to diverge. It should be noted though, that the inclusion of an additional factor will not necessarily cause the output to diverge.

In a final modification of the equation in the first example, the diffusion factor, coefficient f, is changed from .2 to .05. This fourth example is:

$$Y_t = -(.05)Y_{t-1}^2 + (1.0 + (.05)(.8))Y_{t-1} + 0. \qquad (15)$$

or

$$Y_t = -(.05)Y_{t-1}^2 + 1.04Y_{t-1}. \qquad (16)$$

Here A = -.05, B = 1.04, and C = 0. We have

$$K = \frac{-1 \pm \sqrt{1 + 4((\frac{1.04}{2})^2 - \frac{1.04}{2} - 0)}}{-2}$$

or

$$K = \frac{-1 \pm \sqrt{1 - .9984}}{-2} = \frac{-1 \pm \sqrt{.0016}}{-2} = .52 \text{ or } .48.$$

Taking the $K \geq .5$, and once again setting Y_0 equal to .05, the following calculations result:

$$|AY_0 + \frac{B}{2}| = |(-.05)(.05) + \frac{1.04}{2}| = .5175.$$

Thus $.5175 < K$, and Condition IV holds, with Y_n converging to $Y^* = \dfrac{1 - .52 - \frac{1.04}{2}}{-.05} = .8$, as expected.

Table 3 displays the values of the output of Equation (15), and these values approach the limit in this example much more slowly than the output values approach the limit in either of the other two examples with converging time paths. Figure 3 permits a comparison of the time paths for the three converging equations analyzed in this section, and the impact of decreased values for the limit and the diffusion factor (f) are evident.

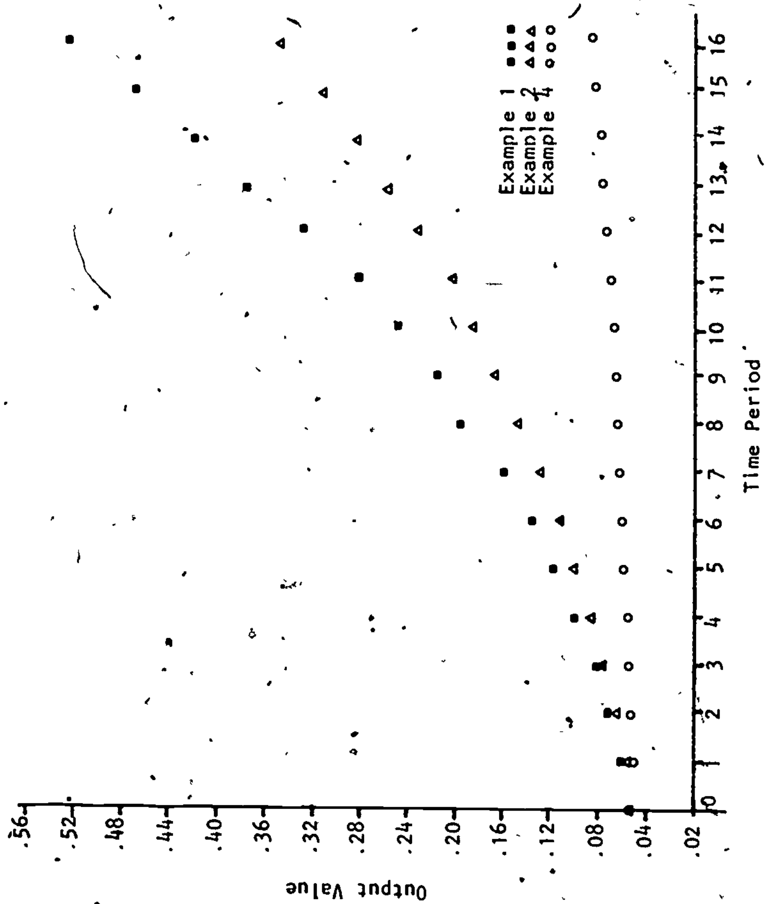


Figure 3. Time Paths for Examples 1, 2, and 4.

TABLE 3

Values of Output from Equation (16)

$$Y_t = (-.05)Y_{t-1}^2 + 1.04Y_{t-1}$$

where $Y_0 = .05$ and the Limit = .8.

Time Period	Value of Y_t
0	.05
1	.0519
2	.0538
3	.0558
4	.0579
5	.0600
6	.0623
7	.0646
8	.0669
9	.0694
10	.0719
11	.0745
12	.0773
13	.0800
14	.0829
15	.0859
16	.0890

EXERCISE 3.1 Given the following equation:

$$Y_t = Y_{t-1} + (.15)Y_{t-1}(1.0 - Y_{t-1})$$

and $Y_0 = .05$

(a) Convert this equation into standard quadratic format, as in Equation (7).

(b) Utilizing the Chaundy and Phillips theorem, identify K and $|AY_0 + \frac{B}{2}|$.

(c) Characterize the time path of this equation, identifying the convergence value if applicable.

EXERCISE 3.2 Given the following equation:

$$Y_t = eY_{t-1} + fY_{t-1}(L - Y_{t-1}), \text{ with } L = 1.0$$

(a) Can you identify a value for f such that $0 \leq Y_t \leq 1$ and Y_t is not monotonic over time? If so, give one numerical example.

(b) How does this finding relate to the theorem of Chaundy and Phillips?

EXERCISE 3.3 Give a substantive interpretation for the conditions in Chaundy and Phillips' theorem which corresponds to your answer in 3.2(a). What interpretations are required for f, L, and Y_0?

4. DIFFUSION OF INNOVATION IN FAMILY PLANNING: AN EMPIRICAL ANALYSIS

The analytical techniques used in the previous section can now be employed with regard to the diffusion of family-planning policies. Equation (3) will be utilized, i.e.,

$$P_t = eP_{t-1} + fP_{t-1}(L - P_{t-1}). \quad (3)$$

Since the number of countries having adopted a family planning program or policy is known for each year between 1960 and 1976, values for the coefficients (e, f, and L) can be estimated. Since no country terminated a family planning program after one was adopted, e can be set equal to 1.0. The output, P_t, can be regressed on P_{t-1} and P_{t-1}^2 to obtain estimates for the coefficient f and the limit L since e is known. The equation used in the regression is

$$P_t = \beta_1 P_{t-1} + \beta_2 P_{t-1}^2 \quad (16)$$

where $\beta_1 = e + fL$

and $\beta_2 = -f$.

The results of the regression are as follows:

$$P_t = (1.37) P_{t-1} + (-.006) P_{t-1}^2 \quad (17)$$

The diffusion factor, f, is equal to .006 and the limit, L, or convergence value, is approximately 62 countries. This finding is interesting, since it indicates that by 1976 just about all countries which are likely to adopt a family planning program or policy have already done so. The remaining pool of potential adopters is thus dry. Due to religious or political pressures, current non-adopters are likely to remain non-adopters. In other words the diffusion process has reached the saturation point, unless other significant factors are introduced, i.e., additional variables in the equations.

Table 4 shows the actual number of countries having adopted family planning programs or policies between 1960 and 1976 and the predicted number, given the values estimated through regression. The two numbers at each time period are clearly quite close. The correspondence between actual and predicted values is further clarified in Figure 4, where the two values are plotted against time.

Applying the Chaundy and Phillips theorem to this equation with empirically derived coefficients shows the following:

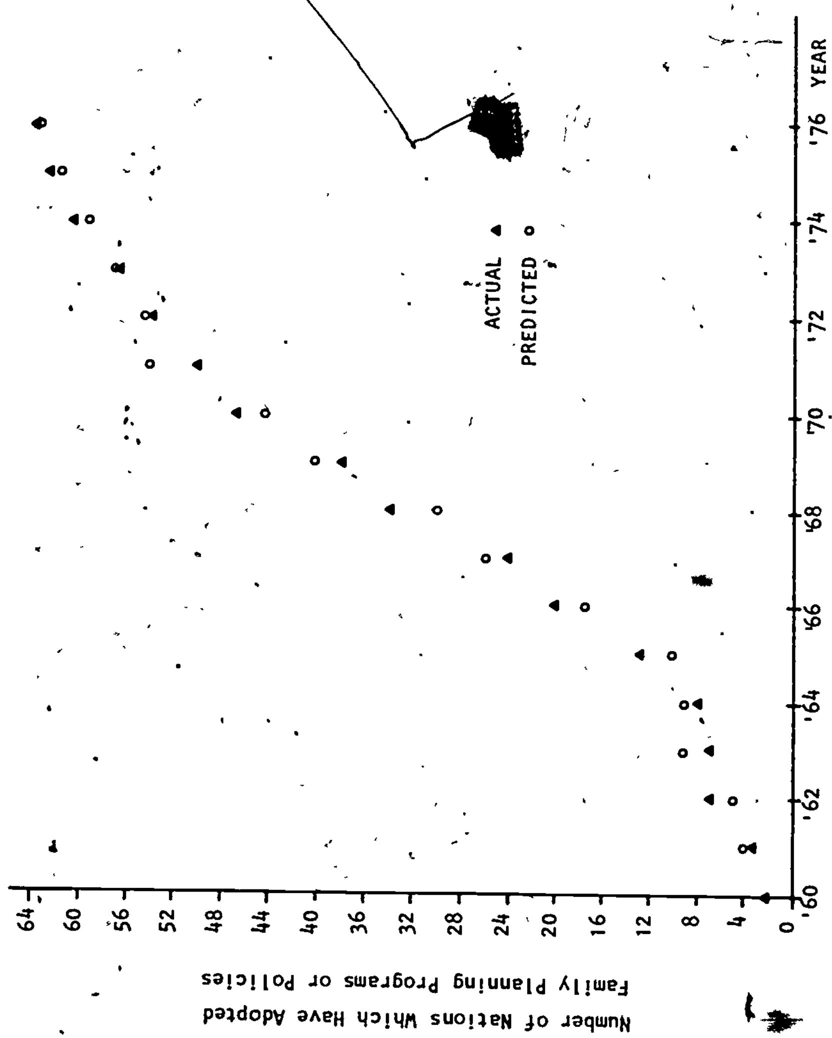


Figure 4. Comparison of Actual Number of Nations Having Adopted Family Planning Programs and Policies, and Predicted Number of Adopters.

TABLE 4

Actual and Predicted Number of Countries with Family Planning Programs or Policies, 1960-1976

Year	Actual Number	Predicted Number
1960	2	
1961	3	3.56
1962	7	4.90
1963	7	10.15
1964	8	10.15
1965	13	11.44
1966	20	17.68
1967	24	25.92
1968	34	30.36
1969	38	40.65
1970	47	44.44
1971	50	54.26
1972	54	54.65
1973	57	58.68
1974	61	59.83
1975	63	62.52
1976	64	63.80

$$K = \frac{-1 \pm \sqrt{1 + 4\left(\left(\frac{1.37}{2}\right)^2 - \frac{1.37}{2} - 0\right)}}{-2}$$

or

$$K = \frac{-1 \pm \sqrt{1 - .8631}}{-2} = \frac{-1 \pm .369}{-2} = .685 \text{ or } .315.$$

With K equal to .685 and an initial condition of 2 countries having adopted family planning programs or policies:

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$$|A^*A_0 + \frac{B}{2}| = |(-.006)(2.0) + \frac{1.37}{2}| = |-.012 + .685| = .673.$$

Since this quantity, .673, is less than K, Condition IV holds, and A_n converges to

$$\frac{1 - K - \frac{B}{2}}{-2}, \text{ or } \frac{1 - .685 - \frac{1.37}{2}}{-2} = 62 \text{ countries, which}$$

is the limit obtained from the regression.

The application of the Chaundy and Phillips theorem to diffusion equations of the sort introduced in this paper can help characterize the diffusion process systematically and alert the researcher to the significance of such things as the size of the pool of potential adopters, the magnitude of the diffusion factor (or intensity of preferences regarding the innovation), and the impact which other factors operating concurrently with the diffusion process can have in facilitating or impeding the diffusion process.

In the policy area discussed here the mathematical analysis helped to answer the questions raised initially about the significant shift in governmental policies over the last two decades. The diffusion equation corresponded quite well with the empirical information about the adoption of policies, and told us how many countries are likely to adopt such policies.

EXERCISE 4. Suppose that time is measured in months, instead of years, for the example of diffusion in family planning analyzed above, i.e.,

$$P_t = eP_{t-1} + fP_{t-1}(L - P_{t-1}) \qquad (3)$$

(a) What happens to the appearance of Figure 3?

(b) What consequence will there be for the parameter f?

(c) Generalize your answer to (b).

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5. REFERENCES

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6. ANSWERS TO EXERCISES

Exercise 1:

(a) $K = \frac{-1 \pm \sqrt{.64}}{-2} = \frac{-1 \pm .8}{-2} = .1,\ .9$

(b) $K = \frac{-1 \pm \sqrt{.64}}{-2} = \frac{-1 \pm .8}{-2} = .1,\ .9$

(c) $K = \frac{-1 \pm \sqrt{.48}}{-2} = \frac{-1 \pm .6928}{-2} = .15359,\ .84641$

Exercise 2:

(a) Given $K = .9$ and $\left|AY_0 + \frac{B}{2}\right| = .1 + .1 = .2$, Condition IV holds, and Y_n converges to $\frac{1 - K - \frac{B}{2}}{A}$ which is $\frac{1 - .9 - \frac{.2}{2}}{1} = .000$.

(b) Given $K = .9$ and $\left|AY_0 + \frac{B}{2}\right| = (.1)(-.2) + .1 = .08$, Condition IV holds and Y_n converges to $\frac{1 - .9 - \frac{.2}{2}}{-.2}$ or .000.

(c) Given $K = .846$ and $\left|AY_0 + \frac{B}{2}\right|$ $(.1)(-.2) + .2 = .08$, Condition IV holds, and Y_n converges to $\frac{1 - .846 - \frac{.2}{2}}{-.2}$ or -.27.

Exercise 3.1:

(a) $Y_t = (-.15)Y_{t-1}^2 + (1.15)Y_{t-1} + 0$

(b) $K = \frac{-1 \pm \sqrt{1 + 4\left(\left(\frac{1.15}{2}\right)^2 - \frac{1.15}{2} - 0\right)}}{-2} = \frac{-1 \pm \sqrt{1 + (-.976)}}{-2}$

$= \frac{-1 \pm \sqrt{.024}}{-2} = .575$ and $.425$

and $\left|AY_0 + \frac{B}{2}\right| = \left|(-.15)(.05) + \frac{1.15}{2}\right| = -.0075 + .575$

$= .5675.$

(c) Since $K = .575$, Condition IV holds and Y_n converges to

$\frac{1 - .575 - \frac{1.15}{.2}}{-.15} = 1.0.$

Exercise 3.2:

(a) Many answers possible.

(b) The f must lead to a K which will give one of the conditions where Y_K oscillates.

Exercise 3.3:

Many answers possible.

Exercise 4:

(a) The time path increases more slowly.

(b) f will be much smaller.

(c) The time frame utilized in your analysis affects parameters and substantive assessment of learning process taking place.

STUDENT FORM 1

Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name ______________________________ Unit No. ________

Page ______ ○ Upper ○ Middle ○ Lower	OR	Section ______ Paragraph ______	OR	Model Exam Problem No. ______ Text Problem No. ______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature ______________________

Please use reverse if necessary.

STUDENT FORM 2

Unit Questionnaire

Return to:
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Name ________ Unit No. ________ Date ________

Institution ________ Course No. ________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

____Much Longer	____Somewhat Longer	____About the Same	____Somewhat Shorter	____Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain ________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain ________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)

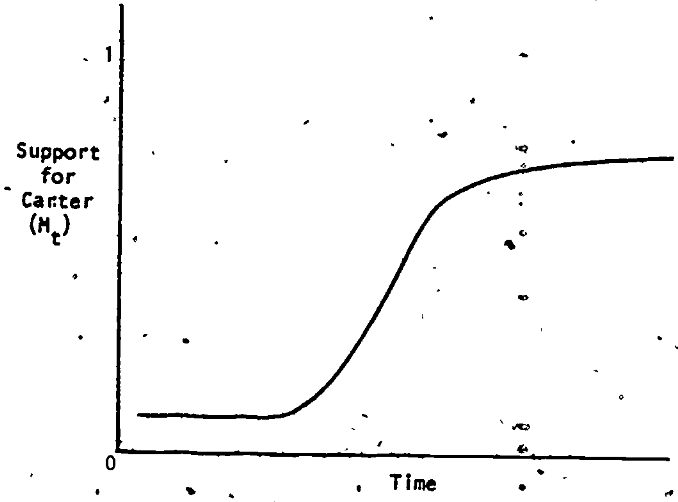
umap

UNIT 304

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

THE GROWTH OF PARTISAN SUPPORT I: MODEL AND ESTIMATION

by Carol Weitzel Kohfeld



APPLICATIONS OF FIRST ORDER QUADRATIC DIFFERENCE EQUATIONS TO AMERICAN POLITICS

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THE GROWTH OF PARTISAN SUPPORT I: MODEL AND ESTIMATION

by

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TABLE OF CONTENTS

Intermodular Description Sheet: UMAP Unit 304

Title: THE GROWTH OF PARTISAN SUPPORT I: MODEL AND ESTIMATION

Author: Carol Weitzel Kohfeld
Department of Political Science
University of Missouri-St. Louis
St. Louis, Missouri 63121

Review Stage/Date: III 1/23/79

Classification: APPL FIRST ORD QUAD DIFF EQ/AMER POL

Suggested Support Materials:

References: See Section 5 of text.

Prerequisite Skills:
1. Understand first order linear difference equations with constant coefficients.
2. Understand the basis for least squares estimation.

Output Skills:
1. Introduction of nonlinear representation (first order quadratic difference equation) for political mobilization processes.
2. Estimation of model parameters and substantive interpretations.
3. Investigating analytic consequences of substantive assumptions.

Other Related Units:
Exponential Models of Legislative Turnover *(Unit 296)*
The Dynamics of Political Mobilization I *(Unit 297)*
The Dynamics of Political Mobilization II *(Unit 298)*
Public Support for Presidents I *(Unit 299)*
Public Support for Presidents II *(Unit 300)*
Laws that Fail I *(Unit 301)*
Laws that Fail II *(Unit 302)*
The Diffusion of Innovation in Family Planning *(Unit 303)*
Growth of Partisan Support II *(Unit 305)*
Discretionary Review by the Supreme Court I *(Unit 306)*
Discretionary Review by the Supreme Court II *(Unit 307)*
What Do We Mean By Policy? *(Unit 310)*

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

This unit was presented in preliminary form at the Shambaugh Conference on Mathematics in Political Science Instruction held December, 1977 at the University of Iowa. The Shambaugh Fund was established in memory of Benjamin F. Shambaugh who was the first and for forty years served as the chairman of the Department of Political Science at the University of Iowa. The funds bequeathed in his memory have permitted the department to sponsor a series of lectures and conferences on research and instructional topics. The Project would like to thank participants in the Shambaugh Conference for their reviews and all others who assisted in the production of this unit.

This material was prepared with the support of National Science Foundation Grant No. SED76-19615 A02. Recommendations expressed are those of the author and do not necessarily reflect the views of the NSF, nor of the National Steering Committee.

THE GROWTH OF PARTISAN SUPPORT:
MODEL AND ESTIMATION

1. INTRODUCTION

Political mobilization can be conceptualized as a dynamic process. Current levels of support for a political cause are related to past levels of support and similarly future levels of political support are related to current levels of support. When these over time changes in support are characterized by gains from those who were not supporters last time and losses from those who were supporters last time we model the process with a first order linear difference equation with constant coefficients. Much is known about first order linear models. Explicit solutions can be determined and conditions for convergence can be specified.*

The mobilization of Carter's support in his drive to become President of the United States provides an example of a qualitatively different form of mobilization. When he first began his campaign, he was relatively unknown and support throughout the United States was low. At first it was difficult to gain supporters from party regulars, but as time went on and he continued to be successful in early primaries and state conventions, supporters began to join the Carter forces at a more rapid rate. Finally Carter's level of support began to level off and new supporters joined him at a much slower rate. Mobilization of Carter supporters can be characterized by a curve with a pronounced S-shape. See Figure 1 for a representation of the process. This is the general form of the demographer's population growth curve: it grows slowly at first, then very rapidly, and finally levels off.

* See Huckfeldt, Robert. "The Dynamics of Political Mobilization I and II." UMAP Units 297 and 298. Also Salert, Barbara. "Public Support for Presidents I and II." UMAP Units 299 and 300.

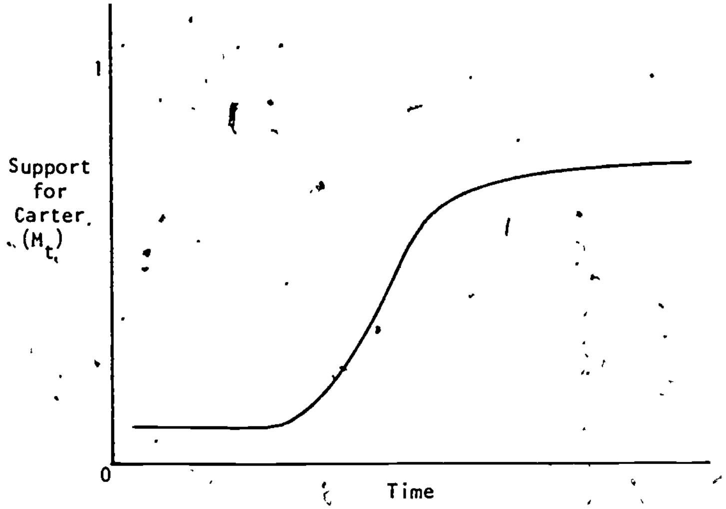

Figure 1. Growth of support for Carter in his campaign as Presidential candidate. (Hypothetical.)

This leads us to ask how we can understand mobilization which takes this form. The growth of support for Carter is similar to processes characterized as contagions. In public health we think of the spread of contagious diseases. When the disease is first introduced into an area it spreads slowly. Then as the number who contract the disease increases, contacts between those who have and those who do not have the disease increase and the disease spreads very rapidly. Finally the number of people left to contract the disease is diminished, the chance of contact between those who have the disease and those who can still contract it is severely reduced and the number of new instances of the disease levels off. Consider also the growth of rumors. When a rumor is first started only a few know about it. The number of those who learn about it grows slowly at first, then very rapidly as contacts between those who know and those who do not know increase, finally the number of those who have knowledge of the rumor levels off. Such processes have been modelled using nonlinear representations, and we propose to study mobilization that takes this form by adapting a variant of standard contagion models.

By making an assumption about the interaction between Carter supporters and those who do not support Carter we can produce a curve of data points with the correct shape, using a form of the nonlinear contagion model. Even though an explicit solution for this nonlinear model is not readily available, the essentials of its qualitative behavior may be determined analytically with a little effort.

2. MODEL

Change in Carter's support over time is the process we seek to understand. We represent mobilization of the population in support of Carter by M_t. M_t gives the proportion of the eligible population which supports Carter at time t. The phenomenon of interest is the time path of Carter support (M_t), i.e., how it grows, declines, or alternately does both. The index t is assumed to range over a sequence of integers measuring time at convenient intervals of say days, weeks, or months.

Changes in voter support for a particular party from one election to the next can be explained in terms of gains and losses. A simple process of diffusion is assumed. Information from a constant source is available to the voters. The party of interest, Democrats, gains support from those who did not support the Democratic party last time at some rate g, and loses support from the party faithful at some rate f. We also argue that not all nonsupporters of the Democratic party are potential supporters. This means there are some Republicans who remain Republicans no matter what the Democratic appeal or what negative information they may obtain about their own party. Thus there is a limit L to the proportion of supporters

that the Democrats can hope to achieve. For our purposes we define the operator Δ as the first difference of M_t, i.e., $\Delta M_t = M_{t+1} - M_t$. The preceding argument is formalized in the following way:

$$(1) \qquad \Delta M_t = g(L - M_t) - fM_t$$

where

- g = rate at which nonsupporters are recruited to supporters
- f = rate at which supporters defect and become nonsupporters
- L = the natural limit toward which the Democratic support moves
- M_t = the proportion of the population mobilized in a particular (Democratic) partisan support.

By disaggregating ΔM_t in (1) and rearranging it into the following form

$$(2) \qquad M_{t+1} = (1 - g - f)M_t + gL$$

it can be seen that the mobilization process has been modelled with a first order linear difference equation with constant coefficients. A solution for this equation is available and theorems about first order linear difference equations can be utilized to determine its qualitative behavior. (Solutions and discussion of this model are in previous modules, Huckfeldt, UMAP Units 297 and 298; Salert, UMAP Units 299 and 300.)

Now let us extend the argument about mobilization to include the effects resulting from over time interaction between those who behave in a particular partisan way and those who do not. Using the Carter example we want to include the contextual effects of Carter supporters interacting with those who did not support Carter. Certainly many Carter supporters were effective in convincing nonsupporters that Carter would win the Democratic party's nomination and thus should have their support. Other supporters in their zealousness turned potential supporters away. Nevertheless during the primary campaigns after a slow start, the former were more numerous than the latter and the Carter forces experienced rapid growth in numbers of supporters until after the convention when Carter's support leveled off as it approached its upper limit. We want to model these effects of political context (here, the level of mobilization achieved), on the rate of change of mobilization. The S-curve of the time path in the Carter example leads to the use of a quadratic (nonlinear) but still first order difference equation with constant coefficients. Although we cannot provide an explicit solution for this model we can determine its qualitative behavior analytically.

We can use some hypothetical numbers to illustrate how the gradient changes over time. Assume that .8 of the population are potential Carter supporters. Early in the mobilization process suppose Carter's support is at the .05 level. This leaves .75 of the population left to be recruited. In the absence of a better rule one simply assumes that the probability of an encounter between supporters and nonsupporters in a unit of theoretical time is proportional to the frequency of supporters and nonsupporters. Thus to estimate the probability of an interaction one multiplies the proportion of supporters and nonsupporters together to obtain the estimate of the likelihood of an interaction occurring. This amounts to assuming that the populations of supporters and nonsupporters interact or mix randomly. Indeed the product of two such frequencies or proportions is the classic formulation of an assumption of random mixing in population diffusion or contagion models. (See Rappaport in Luce et al., 1963.) The product of our assumption yields the probability of an interaction in our theoretical unit of time approximately equal to .04. One can think of this as specifying a time slope or derivative of the process at

a point in time. But if the process is indeed contagious, that is, if mobilization is occurring, the number of supporters is increasing. Suppose the number of supporters has reached the .4 level and recalculate the probability of a substantively significant interaction in our theoretical unit of time. Now we have .16 as our estimate of the time gradient of the process at that point. The process is now growing at four times the rate it was growing at the earlier time point. Finally suppose the process has reached the level of .7. A similar calculation at this last time point shows that the time gradient is now .07 or less than one-half the gradient at the midpoint of .4. Note if these three numbers were drawn as vectors at three equally spaced time points say 10 units apart, they provide the crude outline of the smooth S-curve we are seeking. An exercise on the random mixing assumption occurs after the model is introduced.

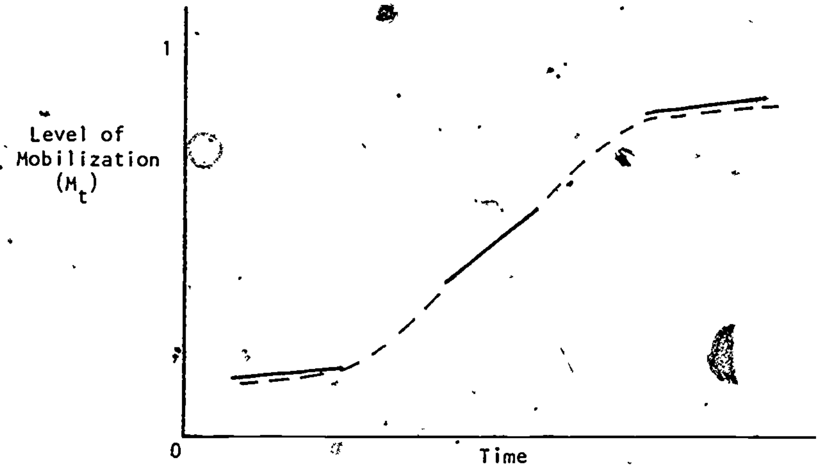

Figure 2. Time gradients of the interaction process as an approximation of the S-curve.

The extended model to include interaction takes the following form. We assume that the rate of change specifies a simple diffusion process. Thus there is a loss (defection) term, a gain (recruitment) term which can be interpreted as spread from a constant source and operating on the out-population, and finally an interaction term which may have either a positive or negative sign depending on whether the outcome of interaction is to swell or diminish those behaving in the fashion characterized as M_t, in this case support of Carter. The proportion of the population recruited by the means of the constant source effects are "removed" from the population of potential supporters who are involved in interaction with the Carter supporters. This argument is formalized generally as follows:

$$(3) \qquad \Delta M_t = g(L - M_t) - fM_t + s[(L - M_t) - g(L - M_t)]M_t.$$

We have added an interaction term to the gain/loss model (1). In words, using the Carter example, the rate of change in mobilization of support for Carter is proportional to potential supporters at rate g, is disproportional to present supporters at rate f, and is affected by the interaction between Carter supporters and recruitable nonsupporters at rate s. The sign of s can be determined empirically, or substantively on other grounds, or it can be set theoretically. In the case of Carter support we can set the sign of s positive and we know the initial value M_0 is low relative to the potential level. L sets the limit of the potential supporters for Carter.

This model turns out to be nonlinear in its parameters (f, g, L, and s) but not critically so if we can impose an a priori hypothesis about L. The easiest solution to the problem is to set L = 1, that is, make the entire population eligible for recruitment to Carter's camp. If we set L at a particular value, the model is still nonlinear in parameters but now a solution is possible. If we

disaggregate ΔM_t and use algebraic manipulation, we can put model (3) into the following form:

$$(4) \qquad M_{t+1} = (sg - s)M_t^2 + (1 + sL - f - g - sgL)M_t + gL.$$

This is equivalent to the least squares estimating form

$$(5) \qquad Y = m_2X_2 + m_1X_1 + m_0$$

where

$$Y = M_{t+1}$$
$$X_2 = M_t^2$$
$$X_1 = M_t.$$

It is instructive to note that the only observations required for estimation of this model are time series of <u>single</u> variables. If we use the Carter example, we need a time series of Carter's support throughout his campaign for the nomination. Other examples are voting data for a particular party in particular urban areas, counties, or minor civil divisions over a period of years. Using a time series of a single political variable we can investigate the relative contributions of individual level (g and f) and contextual (s) effects within the same model. Least squares procedures may be used to obtain estimates of the coefficients m_i in (5). The estimating relationships for the parameters are specified by the system

$$(6) \qquad \begin{aligned} m_2 &= sg - s \\ m_1 &= 1 + sL - f - g - sgL \\ m_0 &= gL. \end{aligned}$$

Let us assume that all of the population are potential supporters and set L = 1. With this assumption system (6) can be solved for the parameters s, g, and f in terms of the coefficients m_i. This is left for the reader as an exercise.

Exercise 1. Set L = 1 and solve system (6) for s, g, and f in terms of m_i.

Exercise 2. Find the point at which the process is growing most rapidly when the upper limit L equals respectively 1, .8, .5 with f and g = 0.

Even though it is relatively easy to obtain values for the parameters using least squares estimation procedures, an explicit solution for the quadratic form of the first order difference equation is not available. We can however determine its qualitative behavior. With estimates for the parmeters and initial conditions particular histories can be generated using the recursive form in (4).

3. THREE EXAMPLES

We now apply the model to three time series of voting data for the Democratic party in three different counties. The time period examined is 1920-1972 and the vote used in each of the three counties is the Democratic vote for president. (Data collected from America At The Polls, and America Votes, edited by Richard Scammon.) The three counties used are Essex County, Massachusetts; Wayne County, Michigan; and Dupage County, Illinois.

What we want to know is how strong were the forces working in the Democratic direction and away from the Democratic direction for individual effects and how strong was the impact of interaction between citizens. The quantities f, g and s give us a hint about the answer to these questions. The parameters f and g give individual level results while s is interpreted as giving contextual effects from interaction. For reasons of descriptive adequacy we restrict f and g to the 0,1 interval. Substantively this means we cannot produce more supporters from supporters nor more nonsupporters from nonsupporters.

The parameter s however is allowed to range over the -1,1 interval and is determined empirically. This allows the contextual effects of mixing with the out-group to either swell or diminish the ranks of the in-group or supporters.

Parameter estimates for Essex County are presented in Table 1. The set of estimates displayed were calculated

TABLE 1

Parameter Estimates for Essex County, Massachusetts, Democratic Presidential Voting, 1920-1972.

Parameter	Description	L = 1
f	individual level loss	.34
g	individual level gain	.16
s	interaction (contextual) effect	.54

$$M_{t+1} = (sg - s)M_t^2 + (1 + sL - f - g - sgL)M_t + gL$$

$$(m_0 = .16,\ m_1 = .95,\ m_2 = -.45)$$

from the m_i setting L = 1. The reader should be sure that he or she can calculate f, g, and s from the least squares estimates of the m_i reported in Table 1.

Exercise 3. Set L = .74 and calculate f, g, and s for Essex County. What does it mean substantively to set L = .74?

The parameter values indicate that contextual level effects are stronger than individual level effects. Essex County has a strong Democratic organization which suggests that gains by the Democratic party are a result of interactions between Democratic supporters and recruitable nonsupporters. The individual level effects are also important but less than the contextual effects. This suggests that information from national sources is having an impact on individual voters in the county. National level activity is inducing individual level change at the local level.

When we set L = .74 we are saying that there is a hard core of Republicans and others who are unaffected by Democratic appeal either from the national level or from interaction with local Democrats. Choosing to set L equal to .74 seems reasonable because this was the highest percentage of Democratic votes for President during the time period considered. Notice the change in the parameter estimates when L is set to .74. Contextual level effects are increased slightly but individual loss effects are reduced. This provides support for earlier studies (Berelson et al., 1954; McPhee and Glaser, 1962) which indicated the importance of personal contact in mobilizing supporters.

Note that although we restricted the parameters f, g, and s the values of the m_i were empirically determined. One test of whether the model provides a reasonable explanation is to examine the parameters. If the parameters meet the conditions of descriptive adequacy then this is persuasive evidence for using the model to explain the mobilization process.

Another test of the model is to compare the observed time path with the predicted time path using the model. These time paths for Essex County are displayed in Figure 3. The shape of the curve appears identical to those we observed for the first order linear model. This appears peculiar for our argument anticipates an S-shaped curve. The reason is that the time series is truncated on the left for the elections prior to 1920 which exhibits low Democratic support. In other words, the portion of the fitted curve that we can catch from the observations available to us only allow us to see the upper tail of the process. Note that the predicted time path captures the change which

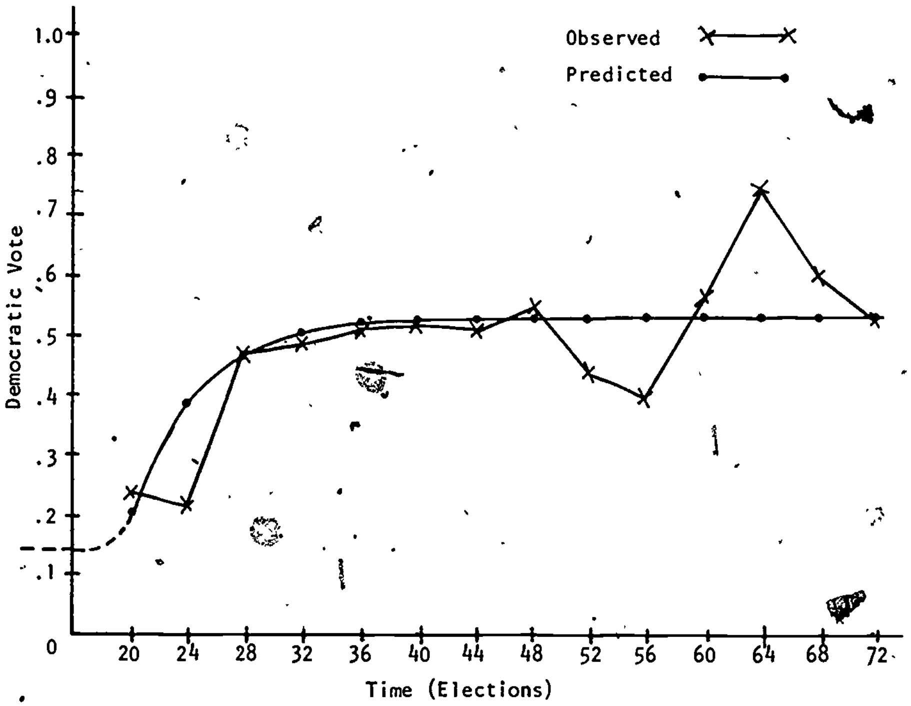

Figure 3. Observed and predicted time paths of Democratic Vote in Essex County, Massachusetts, 1920-1972.

$$\Delta M_t = g(L-M_t) - fM_t + sM_t\,((L-M_t) - g(L-M_t))$$

took place between 1920-1948. During the 1950's the strong national appeal of Eisenhower seems to have counteracted the contextual effects of the local Democratic organization. Similarly, the weak national appeal of Goldwater coupled with the strong national appeal of Johnson account for the short term increase in Democratic vote in 1964. Short-term political forces bump the system but over time it appears to track toward the predicted time path.

Next we want to look at an urban county, e.g., Wayne County (Detroit), which experienced large immigration patterns during the period. It is expected that an even larger contextual effect would be found which can be interpreted as resulting from party organizational efforts in the presence of rapidly changing social composition. The results of estimating the parameters for Wayne County are given in Table 2. Again L is set equal to 1 and the

TABLE 2

Parameter Estimates for Wayne County, Michigan, Democratic Presidential Voting, 1920-1972.

Parameter	Description	L = 1
f	individual level loss	.28
g	individual level gain	.15
s	interaction (contextual) effect	.65

$$M_{t+1} = (sg - s)M_t^2 + (1 + sL - f - g - sgL)M_t + gL$$

$(m_0 = .15,\ m_1 = 1.12,\ m_2 = -.55)$

estimates when L is set to .74 are left to the reader. We set L = .74 for this county and calculate the parameters so that the results can be compared with those obtained for Essex County.

Exercise 4. Calculate f, g, and s for Wayne County using the least squares estimates for the m_i setting L = .74. Explain what difference this makes.

The estimates given in Table 2 are consistent with an organizational contextual effect interpretation--the contextual parameter s is considerably larger than either of the individual level effect parameters. Additional plausibility is gained from the relative magnitudes of f and g. They indicate a higher loss rate if all of the

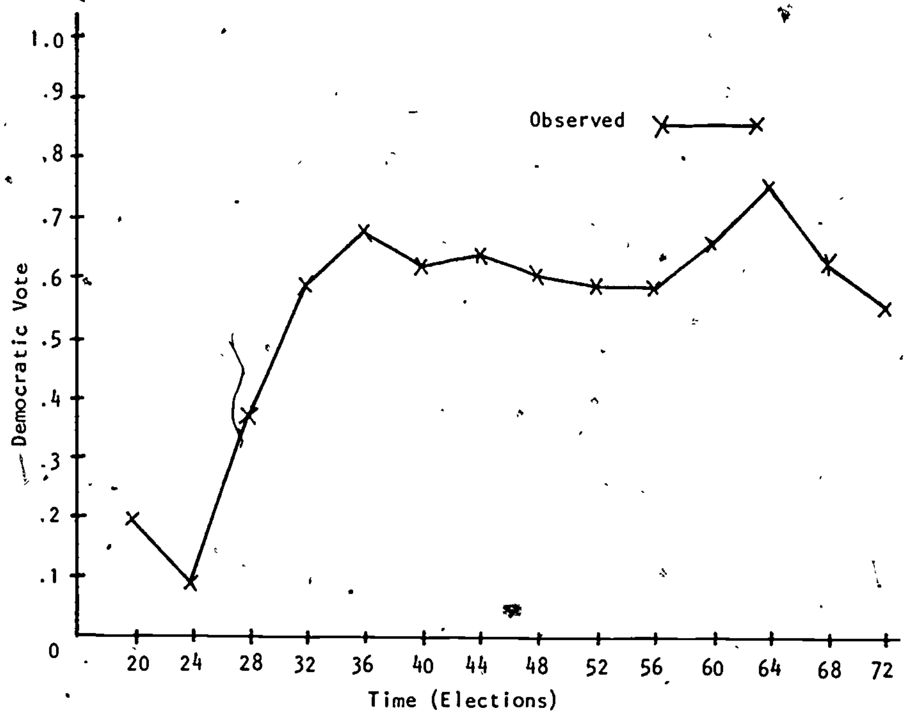

Figure 4. Observed time path of Democratic Vote in Wayne County, Michigan, 1920-1972.

$$\Delta M_t = g(L-M_t) - fM_t + sM_t\,((L-M_t) - g(L-M_t))$$

population is considered as potential Democrats. This loss is apparently recovered by successful organizational effort. When we restrict the potential recruitable population to .74 then the gain rate is larger than the loss rate. This seems to be the more reasonable estimate because throughout most of the period the national level information would reinforce on an individual level what appears to be strong contextual effects in the community.

Again we check the plausibility of the model by examining the size of the parameters and find that each one under both estimates for L meets conditions of descriptive adequacy. The observed time path for the time period is presented in Figure 4. The calculation and plotting of the predicted time path is left for the reader as an exercise.

Exercise 5. Calculate the predicted time path for the Democratic vote using the following recursive form

$$M_{t+1} = (sg - s)M_t^2 + (1 + sL - f - g - sgL)M_t + gL$$

with M_0 = .08; t = 1-7. Plot your results on Figure 4 and evaluate the model.

The sign of s using Democratic presidential vote in Essex and Wayne Counties presented in Tables 1 and 2 was positive. We can illustrate the opposite effect from context by using Democratic presidential voting data from Dupage County, Illinois. Dupage County is a wealthy suburb of Chicago with strong Republican organization.

TABLE 3

Parameter Estimates for Dupage County, Illinois, Democratic Presidential Voting, 1920-1972.

Parameter	Description	L = 1
f	individual level loss	.11
g	individual level gain	.18
s	interaction (contextual) effect	-.57

$$M_{t+1} = (sg - s)M_t^2 + (1 + sL - f - g - sgL)M_t + gL$$

$(m_0 = .18,\ m_1 = .24,\ m_2 = .47)$

Democrats in this county are the minority party. Estimates for the parameters using Dupage County data are displayed in Table 3. The sign of s is negative and its magnitude is larger than either of the individual level effects. These results suggest that as the minority party, Democrats were unable to resist the effects of context, i.e., interaction with the dominant Republican Party in this county. Gains for the Democrats in this county result from national level activity, i.e., individual level effects, but the contextual effects are relatively stronger than the individual ones so as to counteract the national appeal. The model is plausible because the estimates for the parameters meet conditions of descriptive adequacy. The observed time path for Democratic voting in Dupage County is presented in Figure 5. We set L = .5 as well as .74 because Dupage is a Republican County and the Democratic vote never reached higher than the .5 level.

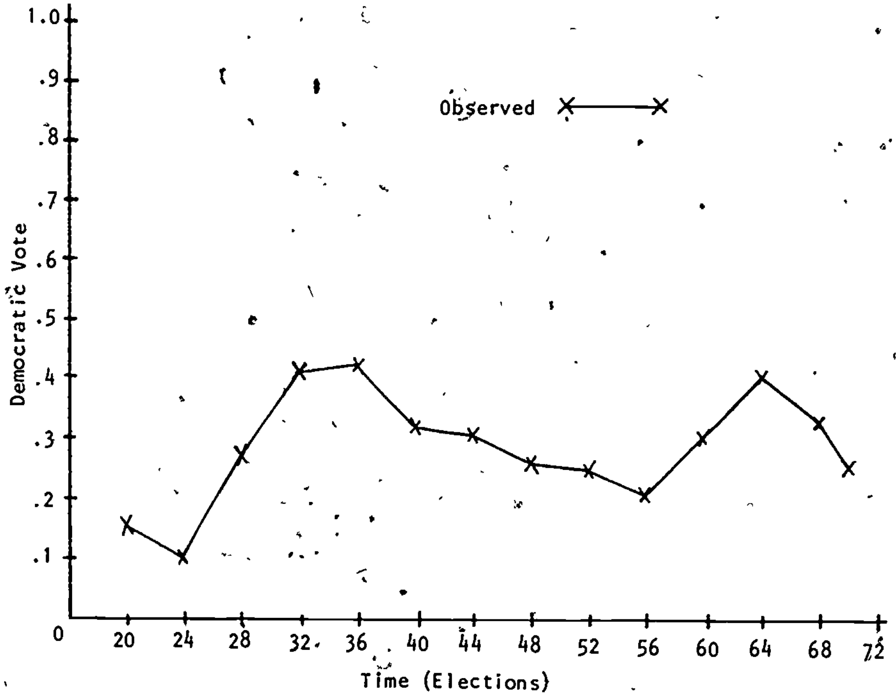

Figure 5. Observed time path of Democratic Vote in Dupage County, Illinois, 1920-1972.

$$\Delta M_t = g(L-M_t) - fM_t + sM_t\,((L-M_t) - g(L-M_t))$$

Exercise 6. Calculate s, f, and g for Dupage County for L = .5 and L = .74. Explain what these results mean.

Exercise 7. Calculate the predicted time path for Dupage County using the recursive form starting at M_0 = .10 with t = 1-7. Plot your results on Figure 5. Evaluate the model for this data.

4. SUMMARY

Political mobilization is conceptualized as a dynamic process. We have extended the simple model of political mobilization which assumes diffusion from a constant source to include contextual effects, in this case political context, resulting from the interaction between supporters for a particular party and recruitable nonsupporters. Addition of the interaction term makes the model nonlinear but not critically so. With the use of least squares estimation and a priori hypotheses about the limit L we can estimate the parameters f, g (individual) and s (contextual). Examination of the parameters and comparison of observed and predicted time paths allow us to evaluate the model as a means for investigating the relative contributions of individual (f and g) and contextual (s) effects. We have applied the model to a single political variable in three counties and found that it provides useful insight into mobilization processes.

The model generates the S-curve characteristic of many self-limiting growth processes. The curve may be pronounced or very gentle. It should be noted that the model exhibits some of the typical properties of nonlinear difference (and differential) equations. For example, convergence depends upon initial conditions which is not the case for linear difference equations. The model is also sensitive to particular values, i.e.; if it started outside a certain range it will diverge, but if moved just a tiny bit it will converge.

It is to limiting behavior and conditions of convergence that we turn our attention next. These behaviors for this model for these three substantive examples are the topic of the next module.

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6. ANSWERS TO EXERCISES

1. .5, .4, .25

2. $f = 1 - m_0 - m_2 - m_1$

   $g = m_0$

   $s = m_2/(m_0 - 1)$

3. f = .17; g = .21; s = .57

   When L = .74 it means there is a hard core of nonsupporters who cannot be recruited.

4. f = .09; g = .20; s = .69

5. $M_0 = .08$, $M_1 = .23$; $M_2 = .38$, $M_3 = .50$, $M_4 = .57$, $M_5 = .61$, $M_6 = .63$, $M_7 = .64$

6. $L = .5$: $s = -.73$; $f = .16$; $g = .36$

$L = .74$: $s = -.62$; $f = .17$; $g = .24$

7. $M_0 = .10$, $M_1 = .20$, $M_2 = .25$, $M_3 = .27$, $M_4 = .27$, $M_5 = .27$,
$M_6 = .27$, $M_7 = .27$

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STUDENT FORM 1

Request for Help

Return to:
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55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name ______________________ Unit No. ______

Page ______		Section ______		Model Exam Problem No. ______
○ Upper	OR	Paragraph ______	OR	Text Problem No. ______
○ Middle				
○ Lower				

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature ______________________

Please use reverse if necessary.

STUDENT FORM 2

Unit Questionnaire

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Name________________________ Unit No.____________ Date____________

Institution________________________ Course No.________________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

____Much Longer ____Somewhat Longer ____About the Same ____Somewhat Shorter ____Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain________________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain________________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)

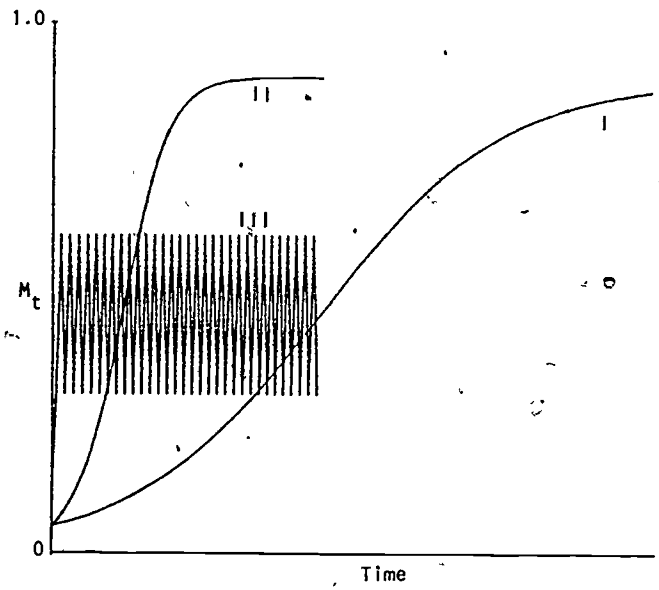
umap

UNIT 305

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

THE GROWTH OF PARTISAN SUPPORT II: MODEL ANALYTICS

by Carol Weitzel Kohfeld



APPLICATIONS OF FIRST ORDER QUADRATIC DIFFERENCE EQUATIONS TO AMERICAN POLITICS

edc/umap 55chapel st newton mass 02160

THE GROWTH OF PARTISAN SUPPORT II: MODEL ANALYTICS

by

Carol Weitzel Kohfeld
Department of Political Science
University of Missouri - St. Louis
St. Louis, Missouri 63121

TABLE OF CONTENTS

Intermodular Description Sheet: *UMAP Unit 305*

Title: THE GROWTH OF PARTISAN SUPPORT II: MODEL ANALYTICS

Author: Carol Weitzel Kohfeld
Department of Political Science
University of Missouri-St. Louis
St. Louis, Missouri 63121

Review Stage/Date: III 1/23/79

Classification: APPL FIRST ORD QUAD DIFF EQ/AMER POL

Suggested Support Materials:

References: See Section 6 of text

Prerequisite Skills:
1. Be able to use first order quadratic difference equations.
2. Familiarity with Taylor series.
3. Familiarity with partial differentiation.

Output Skills:
1. Investigate mathematical properties of first order quadratic difference equation a) equilibria, b) local stability, and c) global stability.
2. Use these analytic results to better understand political mobilization processes.

Other Related Units:
Exponential Models of Legislative Turnover *(Unit 296)*
The Dynamics of Political Mobilization I *(Unit 297)*
The Dynamics of Political Mobilization II *(Unit 298)*
Public Support for Presidents I *(Unit 299)*
Public Support for Presidents II *(Unit 300)*
Laws that Fail I *(Unit 301)*
Laws that Fail II *(Unit 302)*
The Diffusion of Innovation in Family Planning *(Unit 303)*
Growth of Partisan Support I *(Unit 304)*
Discretionary Review by the Supreme Court I *(Unit 306)*
Discretionary Review by the Supreme Court II *(Unit 307)*
What Do We Mean By Policy? *(Unit 310)*

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

This unit was presented in preliminary form at the Shambaugh Conference for Mathematics in Political Science Instruction held December, 1977 at the University of Iowa. The Shambaugh Fund was established in memory of Benjamin F. Shambaugh who was the first and for forty years served as the chairman of the Department of Political Science at the University of Iowa. The funds bequeathed in his memory have permitted the department to sponsor a series of lectures and conferences on research and instructional topics. The Project would like to thank participants in the Shambaugh Conference for their reviews and all others who assisted in the production of this unit.

This material was prepared with the support of National Science Foundation Grant No. SED76-19615 A02. Recommendations expressed are those of the author and do not necessarily reflect the views of the NSF, nor of the National Steering Committee.

THE GROWTH OF PARTISAN SUPPORT: MODEL ANALYTICS

1. INTRODUCTION

In UMAP Unit 304, "The Growth of Partisan Support I: Model and Estimation," we learned that a nonlinear, quadratic, difference equation had useful applications for modeling processes which can be explained by diffusion or contagion. In particular, we found the first order quadratic applicable to the growth of the Democratic party during the Roosevelt revolution and to the more recent phenomenon of Carter's growth of support during his campaign for the presidential nomination. The simple first order linear difference equation model which assumes individual level sources of change was extended to include the effects of political context. Political context is defined as the effects of interaction between those who behave in a particular political way and those who do not and is formalized with the classic formulation of an assumption of random mixing in population diffusion or contagion models. Although the model is quadratic we were able to find estimates for the parameters and to assess the relative contributions of individual and contextual level effects on the mobilization process.

We turn now to the analytic properties of the model. We want to know what the mathematical properties of the model are. In particular we want to investigate its limiting behavior, conditions for convergence, and possible types of qualitative behavior the model can produce. Is the S-curve the only qualitative behavior possible? Because the model is quadratic we cannot determine an explicit solution but we can solve for equilibrium points and examine stability properties using a Taylor series expansion.[3] This provides information about the process

only in a neighborhood of the equilibrium point. Conditions for convergence (global stability) of the quadratic form have been discussed in Chaundy and Phillips (1936) and we will use their results as explicated by Sprague (1969) to further analyze the limiting and convergence behavior of the model. Finally we will look beyond the behavior we obtain in our use of the model to see what possible qualitative behaviors this relatively simple quadratic form can produce. It turns out that very complex behaviors can be produced with this apparently simple quadratic recursive form.

2. THE MODEL

In this model we assume that the rate of change specifies a simple model of diffusion. Thus there is a loss term (f), a gain term (g) which can be interpreted as spread from a constant source and operating on the out-population, and an interaction term (s) which may have either a positive or negative sign depending on whether the outcome of interaction is to swell or diminish those behaving in the fashion characterized as M_t. The proportion of the population recruited by means of the constant source effects are "removed" from the population of potential supporters who are involved in the interaction with supporters. This argument is formalized as follows:

$$(1)\quad \Delta M_t = g(L - M_t) - fM_t + sM_t[(L - M_t) - g(L - M_t)].$$

In words, using the Carter example: the rate of change in mobilization of support for Carter is proportional to potential supporters at rate g, is disproportional to present supporters at rate f, and is affected by the interaction between Carter supporters and recruitable nonsupporters at rate s. L sets the limit of the potential supporters. For reasons of descriptive adequacy we assume the following restrictions:

$$(2) \qquad 0 \le f, g, L, M_t \le 1 \text{ and } -1 \le s \le 1.$$

These are reasonable restrictions. L and M_t are proportions of the population and the restriction keeps us from having substantively meaningless things such as negative populations or more than 100 percent of the population. Restricting f and g positive does not allow supporters to be gained from supporters nor nonsupporters to be produced from nonsupporters. Finally letting s range both positive and negative allows interaction to produce supporters or nonsupporters as a result of contact between groups.

3. SOLVING FOR EQUILIBRIA

Even though it is relatively easy to obtain values for the parameters, an explicit solution for the quadratic form of the first order difference equation is not available. This means we have no closed form for the nth term of the time series, however particular time paths can be determined using the recursive form.

It is possible to solve the quadratic for its equilibrium points. Rearrange the model in Equation (1) so it has the following form:

$$(3) \qquad \Delta M_t = (sg - s)M_t^2 + (sL - f - g - sgL)M_t + gL.$$

If the process is at equilibrium then any successive values of M_t are of equal value, i.e., $M_{t+1} = M_t$. If such a value or values exist they are called stationary values. Thus we want to know what values for M make the process stationary, that is, when is the change in the mobilization process from one time period to the next zero, i.e., what values satisfy $M_{t+1} = M_t$? We can solve Equation (3) for these points using the quadratic formula. Set $\Delta M_t = 0$ and substitute M* (a stationary value) in Equation (3) for M_t. The result is

$$(4) \qquad 0 = (sg - s)M^{*2} + (sL - f - g - sgL)M^* + gL.$$

Recall the quadratic formula:

$$(5) \qquad \frac{(-b \pm \sqrt{b^2 - 4ac})}{2a}$$

where

$$(6) \qquad a = sg - s$$
$$b = sL - g - f - sgL$$
$$c = gL.$$

Solutions for Equation (4) are represented by

$$(7) \qquad M^* = \frac{-(sL-f-g-sgL) \pm \sqrt{(sL-f-g-sgL)^2 - 4(sg-s)(gL)}}{2(sg-s)}$$

We can apply this formula to the results we obtained in UMAP Unit 304 for Essex, Wayne and Dupage Counties. A summary of the estimates for f, g, s, and L for the three counties are reported in Table 1. These estimates will be

TABLE 1

Summary of Parameter Estimates for Essex, Wayne, and Dupage Counties, Democratic Presidential Voting, 1920-1972.

County	Parameters				Least Squares Coefficients		
	L	f	g	s	m_0	m_1	m_2
Essex	1	.34	.16	.54	.16	.95	-.45
Wayne	1	.28	.15	.65	.15	1.12	-.55
Dupage	1	.11	.18	-.57	.18	.24	.47

used in the calculations which follow in the text and exercises. As an illustrative example consider Essex County. Substituting into formula (7) we have

$$(8) \qquad M^* = \frac{(.05) \pm \sqrt{(.05)^2 - 4(-.45)(.16)}}{2(-.45)}$$

which reduces to

$$(9) \qquad M^* = \frac{.05 \pm \sqrt{.2905}}{-.9}$$

$$M^* = .54,\ -.66\ .$$

What does this tell us about the mobilization process? Since M_t substantively represents a proportion of the population which supports the Democratic party, we are interested in values of M* which lie in the 0,1 interval. Values for M* greater than 1 suggest that the process is stationary when more than all the population are mobilized and similarly values less than zero indicate a stationary process with negative populations. Both of these cases are substantively meaningless for our model so we restrict our consideration to the equilibria which lie in the 0,1 interval.

Exercise 1. Calculate M* for Wayne and Dupage Counties using the estimates for f, g, s, and L. In UMAP Unit 304 you calculated estimates for f, g, and s when L was set at a lower value (.74). What value would M* have if you used these second sets of estimates? Why?

In the case of Essex County we find M* = .54. This means that when the process reaches .54, that is 54 percent of the population are mobilized in support of the Democratic party, gains and losses balance and the process is stationary unless perturbed by some external force. It is important to note that M* is a net state. The process continues at M* but the level of mobilization stays the same, i.e., the process remains dynamic but the measure of the state--a number--no longer changes.

A logical question to ask next is what happens when the process is bumped by external forces? In the political process, we are talking about short run political forces such as political scandals and political personalities. These short run forces would shock the system away from the

equilibrium. After these forces have shocked the political system will the process converge toward the equilibrium point or will it diverge? We have solved the quadratic form to determine its equilibria, but we have not yet provided a means to test for stability. Stability can be local or global. Local stability means that within some specified neighborhood of the equilibrium point, if the process is perturbed it will converge toward the equilibrium point. Whereas global stability implies that the process is stable no matter what the perturbation. (For a more complete description of stability see Rosen, 1970; May, 1973.)

First we apply a technique to investigate local stability at the equilibrium points and then we will take advantage of some general results of Chaundy and Phillips (1936), reworked by Sprague (1969), to determine global stability for the specific quadratic form. The technique used to investigate local stability has a wider and general application for more complex nonlinear models. An analysis of small perturbations around the equilibrium point M* begins by writing the perturbed mobilization level as

$$(10) \qquad M_t = M^* + X_t .$$

Here X_t measures a small disturbance to the equilibrium M* within some specified neighborhood. An approximate difference equation for the perturbation measure is obtained by a Taylor expansion of the equation for our original model (3) about the equilibrium point. The Taylor series expansion provides a linearization of the model because the linear term in the expansion dominates the series in a small neighborhood and terms of order 2 and higher can be neglected. The expansion takes the following form

$$(11) \qquad \Delta X_t = aX_t \quad \text{or} \quad X_{t+1} = (1 + a)X_t$$

where a is the partial derivative of ΔM_t with respect to M evaluated at the equilibrium point M* (obtained from Equation (3)).

$$(12) \qquad a = \frac{\partial(\Delta M_t)}{\partial M_t} = 2(sg-s)M^* + sL - f - g - sgL .$$

It measures the mobilization growth rate in the immediate neighborhood of the equilibrium point.

Equation (11) is a first order linear difference equation for which we have an explicit solution. It has the form

$$(13) \qquad X_t = X_0(1 + a)^t$$

where X_0 is the initial small perturbation. The disturbance dies away if $(1 + a)$ lies in the open interval -1,1, diverges if $(1 + a) > 1$ or $(1 + a) \leq -1$, and is constant if $(1 + a) = 1$. Thus the neighborhood stability analysis of the equilibrium point M^* shows the point to be stable if and only if $-1 < 1 + a < 1$, or more simply $-2 < a < 0$.

Now we apply these results to our example the mobilization of the Democratic party in Essex County. First we substitute (12) into (11) to obtain

$$(14) \qquad \Delta X_t = (2(sg-s)M^* + sL - f - g - sgL)X_t .$$

Disaggregating ΔX_t gives

$$(15) \qquad X_{t+1} = (1 + 2(sg-s)M^* + sL - f - g - sgL)X_t$$

where

$$(16) \qquad 1 + a = (1 + 2(sg-s)M^* + sL - f - g - sgL).$$

Evaluating the coefficient of X_t at $M^* = .54$ using the estimates for the parameters s, f, g, and L from Table 1 gives

$$(17) \qquad (1 + a) = (1 - .49 - .05) = .46.$$

Since $(1 + a) = .46$ and is between 0 and 1 we know that the disturbance is monotonically convergent and dies out. This means that the equilibrium $M^* = .54$ for Essex County is locally stable. The mobilization of the Democratic party converges toward .54 of the population which is a

locally stable equilibrium. (This discussion is adapted from May 1973.)

Exercise 2. Do a local stability analysis of the equilibrium points M* calculated in Exercise 1 for Wayne and DuPage Counties.

---

4. GLOBAL STABILITY ANALYSIS

In general there are no techniques for investigating global stability for nonlinear models. We can investigate local stability by linearizing the model with Taylor series expansions around the equilibrium points, but this only provides stability analyses in the small. There are, however, some general results for a particular nonlinear form, the quadratic, reported by Chaundy and Phillips (1936) and further explicated by Sprague (1969). Chaundy and Phillips consider a difference equation of the following form:

$$(18) \qquad M_{t+1} = AM_t^2 + BM_t + C$$

where A, B, and C are real numbers independent of t. We can immediately see that our model is isomorphic to this form. Chaundy and Phillips do not provide an explicit solution but conditions of convergence, divergence, and ultimate qualitative behavior can be developed from their discussion. Only a few of the results are presented here, the inquisitive reader will search out the original source for a complete explication of their results.

First define a quantity K by

$$(19) \qquad K = \frac{-1 \pm \sqrt{1 + 4[(B/2)^2 - B/2 - AC]}}{-2}$$

where A, B, and C are from Equation (18). This produces 3 possibilities: 2 real and unequal K's, 2 real, equal K's, or a pair of complex K's. Six cases are considered below.

Case I. If K given by (19) is complex then the process is divergent, diverging to infinity.

Now choose that K which is $\geq .5$. One of the K's should meet this condition.

Case II. If $|AM_0 + B/2| > K$ then the process M_t diverges to infinity.

Case III. If $|AM_0 + B/2| = K$ then the process M_t is stationary. This does not mean the process will converge if displaced.

Case IV. If $|AM_0 + B/2| < K$ and $1/2 \leq K \leq 3/2$ then the process M_t converges to a value

$$M^* = \frac{(1 - K - B/2)}{A}. \tag{20}$$

The limit M^* is dependent on A, B, and C since K depends on C. Convergence in this case is monotonic if $1/2 \leq K \leq 1$.

Case V. If $|AM_0 + B/2| < K$ and $3/2 \leq K \leq 2$ then the process M_t oscillates finitely.

Case VI. If $|AM_0 + B/2| < K$ and $K > 2$ then the process M_t diverges to infinity with a certain exception, i.e., if M_0 is chosen so that the expression $AM_0 + B/2$ is an element of a set involving the square roots of the expression $K^2 - K$ then the process M_t oscillates finitely. (This discussion is based on results from Sprague, 1969.)

We now apply these conditions for convergence to the data on Democratic mobilization for Essex County. We have already determined that there is a locally stable equilibrium at $M^* = .54$. We now take advantage of the preceding results to see if the locally stable equilibrium satisfies conditions for global stability. First the real numbers A, B, and C are defined

$$\begin{aligned} A &= sg - s \\ B &= 1 + sL - f - g - sgL \\ C &= gL. \end{aligned} \tag{21}$$

Substituting the estimates for the parameters for s, f, g, and L for Essex County from Table 1 into the formulas in (21) we obtain the values $A = -.45$, $B = .95$, and $C = .16$. Using these values we calculate K as follows

$$K = \frac{-1 \pm \sqrt{1 + 4(.23 - .475 + .07)}}{-2} \tag{22}$$

$$K = .78,\ .23.$$

We choose $K = .78$ as the value for K and find that Case IV applies. Now examine the value $|AM_0 + B/2|$. Substituting values for A and B we obtain $|-.45M_0 + .475|$. The condition for convergence of the process is

$$|-.45M_0 + .475| < .78. \tag{23}$$

Recall that convergence for nonlinear difference equations is dependent upon initial conditions. Thus the starting point of the mobilization process is an important consideration in the determination of long run limiting behavior. For what values of M_0 does the inequality in (23) hold? We begin by looking at the extreme values for M_0. If it holds for the extreme values then it holds for all values of M_0. M_0 can range across the 0,1 interval. Both extreme values 0 and 1 for M_0 satisfy the inequality thus any permissible starting value satisfies the condition for convergence. We also note that convergence of the process is monotonic since K lies in the 1/2,1 interval. Finally, M^* calculated using Equation (20) equals .54 for Essex County. This is the same value obtained using the quadratic formula which is as it should be.

Exercise 3. Use these results to perform a global stability analysis for Wayne and Dupage Counties. Compare the M* you calculate with the M* you calculated using the quadratic formula.

In summary, then, we have investigated the mathematical properties of the first order quadratic difference equation used to model mobilization processes characterized by diffusion or contagion. Although explicit solutions for the quadratic are not available, the quadratic can be solved for equilibrium points using the quadratic formula. Local stability was investigated using a Taylor series expansion around the equilibrium point. But this provides information about stability only in the small, in specified neighborhoods of the equilibrium. In general, for nonlinear models local stability can be investigated using this technique. However, in the case of the quadratic, some general results are known and conditions for convergence and divergence were reported and used to investigate global stability for the mobilization process.

This really is not the end of the usefulness of this simple mathematical form because it is capable of producing rather remarkable behaviors. In the next section we briefly illustrate some of the more dramatic time paths which are produced for arbitrary assignments to the parameters. Although there is no substantive interpretation for most of the behaviors they do suggest that some very complex behaviors which appear to be random or stochastic may be generated by this relatively simple quadratic form.

5. A LOOK BEYOND

We want to examine some of the possible qualitative behaviors which can be produced by the recursive form of the quadratic difference equation. We have seen the simple S-curve produced, but there are many other time paths which can be produced which exhibit bounded behavior and which are much more complex. To illustrate these behaviors we use the simple logistic form of the model (used in Harmon's module, UMAP Unit 303, 1978). The model is formalized as follows

(24) $$\Delta M_t = rM_t(L - M_t)$$

where

r = the intrinsic growth rate of the process, typically a species population

L = the natural limit of the growth process in the population

M = the proportion of the population which behaves in a specified manner or species number.

We want to know what interesting behaviors can be obtained by driving this model with assignments of arbitrary growth rates. In particular what kind of behavior is produced when we drive the model by assigning growth rates which exceed unity? First put the model into the recursive form

(25) $$M_{t+1} = -rM_t^2 + (1 + rL)M_t .$$

Using this form particular time paths can be generated by varying the growth rate and the initial conditions. In Figure 1 three time paths are exhibited: two are the familiar S-curve and the third oscillates with period two. All three are generated by the recursive form in (25). Specific parameters are presented in Table 2.

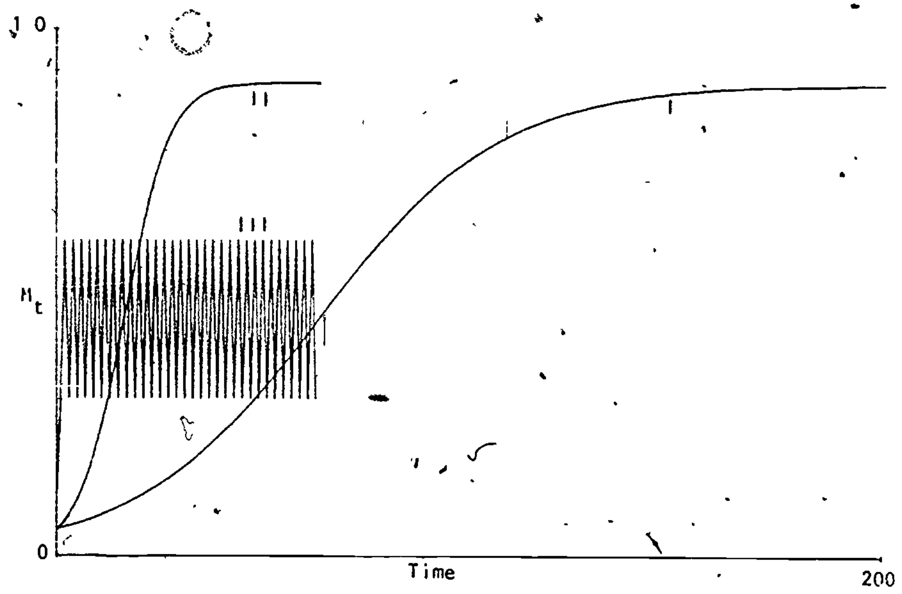


Figure 1. Time paths generated by the simple logistic form: $\Delta M_t = rM_t(L-M_t)$. (Parameters exhibited in Table 2.)

TABLE 2

Parameters for Time Paths Exhibited in Figures 1 and 2 Generated by the Logistic Form $\Delta M_t = rM_t(L - M_t)$

Figure	Time Path	Parameters		Initial Condition
		r	L	M_0
1	I	.05	.9	.1
1	II	.2	.9	.1
1	III	5	.5	.1
2	I	6	.5	.1

This appears orderly but note the departure from smooth growth for the process when the growth parameter r is set greater than 1. But even more is possible. In Figure 2 we exhibit a process generated by the same

recursive form in (25) with r = 6, L = .5, and M_0 = .1. . . Increasing r makes the process more reactive and produces wild oscillation. But notice even this wild oscillation is bounded behavior. Much still has to be learned about the possible behaviors produced by this simple deterministic rule. Looking at the time path in Figure 2 it is hard to imagine experiencing this process as deterministic. (See Li and Yorke, 1975; May, 1973; May, 1974; May, 1975; May, 1976; and May and Oster, 1976.)

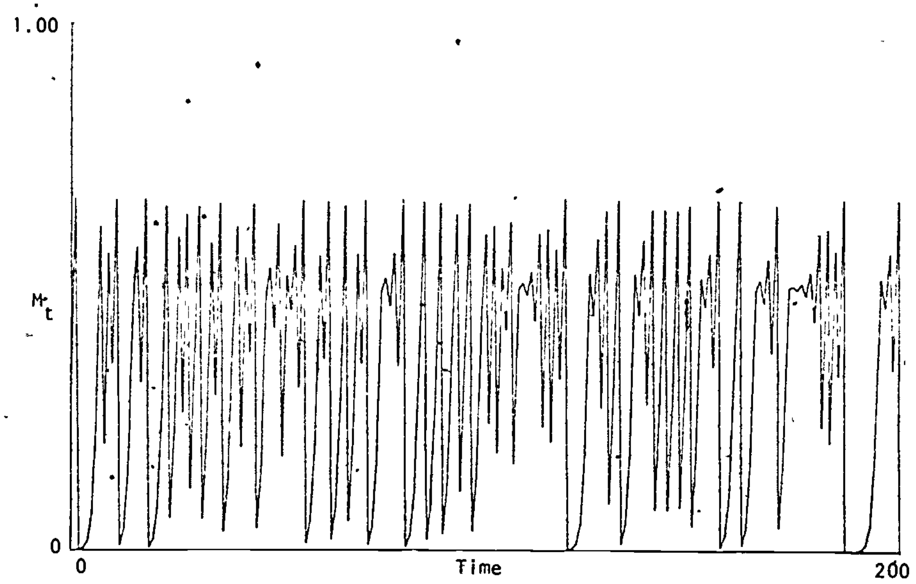


Figure 2. Time path generated by simple logistic form: $M_t = rM_t(L-M_t)$. ($r = 6$, $L = .5$, $M_0 = .1$)

6. REFERENCES

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7. ANSWERS TO EXERCISES

1. Wayne County

$$M^* = \frac{-.12 \pm \sqrt{(.12)^2 + 4(.55)(.15)}}{-2(.55)}$$

$M^* = .64, -.423.$

Dupage County

$$M^* = \frac{.76 \pm \sqrt{(.76)^2 - 4(.47)(.18)}}{2(.47)}$$

$M^* = .29, 1.33.$

You get the same values for M^* with the other estimates for the parameters.

2. Wayne County

$(1 + a) = (1 - 2(.55)(.64) + .12) = .416.$

$.416 < 1$. Disturbance is monotonically convergent and dies away. The process is locally stable.

Dupage County

$(1 + a) = (1 + 2(.47)(.29) - .76) = .51.$

$.51 < 1$. Disturbance is monotonically convergent and dies away. The process is locally stable.

3. Wayne County

$K = .79, .22$ $\qquad M^* = \frac{1 - .79 - .56}{-.55} = .64$

$|-.55M_0 + .56| < .79$

This inequality is satisfied for all permissible values of M_0 therefore the process is globally stable. K lies in 1/2,1 interval, therefore the process is monotonically convergent.

Dupage County

$K = .74,\ .26$

$$M^* = \frac{1 - .74 - .12}{.47} = .29$$

$|.47M_0 + .12| < .74$

This inequality is satisfied for all permissible values of M_0 therefore the process is globally stable. K lies in 1/2,1 interval, therefore the process is monotonically convergent.

80

STUDENT FORM 1

Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name________________________ Unit No.________

Page_____ ○ Upper ○ Middle ○ Lower	OR	Section_____ Paragraph_____	OR	Model Exam Problem No._____ Text Problem No._____

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature________________________

Please use reverse if necessary.

Return to:
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STUDENT FORM 2

Unit Questionnaire

Name__________________________________Unit No._____________Date_____________

Institution_______________________________Course No.________________________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

Much	Somewhat	About	Somewhat	Much
____Longer	____Longer	____the Same	____Shorter	____Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain____

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain____

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)